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THE LOCALIZATION OF ATTRACTORS OF THE LIÉNARD EQUATION†

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A method for localizing the attractors of the Liénard equation is proposed, based on the construction of special piecewise-linear discontinuous comparison systems. © 2002 Elsevier Science Ltd. All rights reserved.

In the past decade, there has been renewed interest in the classical problem of the effective localization of attractors of the Liénard equation and its different generalizations [1–3]. It should be noted that the traditional approach to the localization of the attractors of the Liénard equation [4–6], based on the use of the direct Lyapunov method, is fairly time consuming. The localization estimates obtained by this means prove to be too crude, on account of which only the fact of the existence of global attractors is established.

The development of methods for localizing the attractors of the Liénard equation, begun earlier [7–9], is continued below, based on the construction of special piecewise-linear discontinuous comparison systems. Such discontinuous systems have been studied [10] in relation to the problem of flutter. On the other hand, the introduction of these systems as comparison systems makes the proof of localization theorems considerably simpler than in existing schemes [4–6], and on the other hand, for the van der Pol equation, the approach proposed yields better estimates of the "amplitude" of the limit cycle than those obtained by methods proposed earlier [1–3]. The universality of the construction of the comparison systems examined in the present paper enables us to introduce different variable parameters which improve localization theorems [7–9].

We consider the system

$$dy/dt = -\mu[F(y) - E(t)] - x, \quad dx/dt = y \tag{1}$$

where $F(y)$ and $E(t)$ are functions satisfying the Lipschitz condition, and μ is a positive number.

Below it will be assumed that, for certain positive numbers α and k, the inequality

$$(F(y) - E(t))/y > (\alpha y - k \operatorname{sign} y)/y, \quad \forall t \in \mathbf{R}^1, \quad \forall y \neq 0 \tag{2}$$

is satisfied. Assumption (2) is fairly natural and traditional for the Liénard system (1) [4–6].

We will first consider the case when $\alpha\mu \geqslant 2$. Here, we will introduce the positive half-trajectory of the system

$$dy/dt = -\mu\alpha y - x + \mu k, \quad dx/dt = y \tag{3}$$

with the initial data $y(0) = 0, x(0) = -\mu k$, and the positive half-trajectory of the system

$$dy/dt = -\mu\alpha y - x - \mu k, \quad dx/dt = y \tag{4}$$

with the initial data $y(0) = 0, x(0) = \mu k$.

The solutions $G_1(x, \mu k)$ and $G_2(x, \mu k)$ of the respective first-order equations

$$GdG/dx = -\mu\alpha G - x + \mu k \tag{5}$$

$$GdG/dx = -\mu\alpha G - x - \mu k \tag{6}$$

correspond to these trajectories.

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Below, we will examine the set

$$\Omega(\alpha,k)=\{x\in[-\mu k,\mu k],\quad G_2(x,\mu k)\leqslant y\leqslant G_1(x,\mu k)\}$$

When $\alpha\mu < 2$, we will introduce the set $\Omega(\alpha, k)$ in a slightly different way. It is well known [10] that the system

$$dy/dt=-\mu\alpha y-x+\mu k\,\mathrm{sign}\,y,\quad dx/dt=y \tag{7}$$

has a single limit cycle $x(t), y(t)$ with the initial data $x(0) = \rho, y(0) = 0$. Here

$$\rho=\frac{1+\exp(-\lambda\pi/\omega)}{1-\exp(-\lambda\pi/\omega)}\mu k,\quad \lambda=\frac{\alpha\mu}{2},\quad \omega=\sqrt{1-\lambda^2}$$

The solution of discontinuous system (7) is determined as by Filippov [11]. The solution $G_1(x, \rho)$ of Eq. (5) corresponds to the part of the limit cycle that is positioned in the phase half-space $\{x \geqslant 0\}$. The solution $G_2(x, \rho)$ of Eq. (6) corresponds to the part of the limit cycle that is positioned in the half-space $\{x \geqslant 0\}$.

Remember that the set

$$\Phi(t,t_0)=\begin{Vmatrix} x(t,t_0,\Phi_0) \\ y(t,t_0,\Phi_0)\end{Vmatrix},\quad \Phi(t_0,t_0)=\Phi_0$$

is termed globally attracting if, for any numbers $\varepsilon > 0, x_0 \in \mathbb{R}^1, y_0 \in \mathbb{R}^1$ a number $T(\varepsilon, x_0, y_0)$ exists such that the solution $x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)$ is contained in the ε-neighbourhood of the set $\Phi(t, t_0)$ when $t \geqslant T(\varepsilon, x_0, y_0)$.

We will call the set $\Phi(t, t_0)$ the minimum globally attracting set if there is no true subset of $\Phi(t, t_0)$ that possesses the property of global attraction.

We will refer to the minimum globally attracting set $\Phi(t, t_0)$ of the system (1) as the global attractor of this system.

We will define the set

$$\Omega(\alpha,k)=\{x\in[-\rho,\rho],\quad G_2(x,\rho)\leqslant y\leqslant G_1(x,\rho)\}$$

Theorem 1. The global attractor $\Phi(t, t_0)$ of system (1) is contained in the set $\Omega(\alpha, k)$.

Proof. Let inequality (2) be satisfied for a certain $\alpha > 0$ and $k = k_0$. It is obvious that it is also satisfied for all $k \geqslant k_0$. However, then, for any point (x_0, y_0) of the set $\mathbb{R}^2 \setminus \Omega(\alpha, k_0)$, a number $k \geqslant k_0$ exists such that (x_0, y_0) belongs to the limit $\Omega(\alpha, k)$. Thus, we have a family of closed curves covering the set $\mathbb{R}^2 \setminus \Omega(\alpha, k_0)$.

We will show that all these curves, with the exception of the points $\{y = 0, x \in \mathbb{R}^1\}$, are contactless everywhere and that the trajectories of system (1) "pierce" these curves from outside in. For this, we will use the Chaplygin–Kamke comparison principle [12–15] and inequality (2). We obtain

$$dy/dx=(-\mu(F(y)-E(t))-x)/y<(-\mu(\alpha y-k\,\mathrm{sign}\,y)-x)/y,\quad \forall x\in\mathbb{R}^1,\quad \forall y\neq 0,\quad \forall t\in\mathbb{R}^1$$

From the comparison principle it follows that the solutions $G_j(x)$, corresponding to the trajectories of the system

$$dy/dt=-\mu(\alpha y-k\,\mathrm{sign}\,y)-x,\quad dx/dt=y$$

and the solution $y(t), x(t)$ of system (1) possess the following property at the point $t = t_0, x_0 = x(t_0)$, $y_0 = y(t_0) = G_j(x_0) \neq 0$

$$dy/dx<dG_j/dx$$

The required contactlessness of the curves $y = G_j(x)$ in relation to the vector field of system (1) follows from this, Likewise, the assertion of the theorem follows from the contactlessness of this family of curves nearly everywhere.

Corollary 1. The global attractor of system (1) is contained in the set

$$\Omega_0 = \bigcap_{\alpha,k} \Omega(\alpha,k)$$

where intersection is taken for all parameters α and k satisfying condition (2).

We will now assume that, instead of inequality (2), the following conditions are satisfied

$$(F(y)-E(t))/y > (\alpha y - k\,\mathrm{sign}\,y)/y, \quad \forall t\in\mathbb{R}^1, \quad \forall y\in\{|y|\geqslant\gamma\} \tag{8}$$

$$(F(y)-E(t))/y > \nu\,\mathrm{sign}\,y/y, \quad \forall t\in\mathbb{R}^1, \quad \forall y\in\{|y|\leqslant\gamma\} \tag{9}$$

Where ν and γ are certain numbers, $\nu < 0$. In this case, instead of comparison equations (5) and (6), we introduce the following equation:

$$F dF/dx = -f(F) - x \tag{10}$$

$$f(F) = \begin{cases} \mu\alpha F - k\mu, & F\geqslant\gamma \\ \mu\nu, & F\in(0,\gamma) \\ -\mu\nu, & F\in(-\gamma,0) \\ \mu\alpha F + k\mu, & F\leqslant-\gamma \end{cases}$$

We will examine the solutions $F_1(x)$ and $F_2(x)$ of Eq. (10), positioned in the half-planes $\{F \geqslant 0\}$ and $\{F \leqslant 0\}$ respectively with the initial data $F_i(\rho) = F_i(-\rho) = 0$ $(i = 1, 2)$. Here ρ is a certain number.

It is clear that these solutions correspond to the limit cycle of the system

$$dy/dt = -f(y) - x, \quad dx/dt = y$$

with the initial data $y(0) = 0, x(0) = \rho$. It can be shown that such a limit cycle is unique.

Theorem 2. The global attractor $\Phi(t, t_0)$ of system (1) is contained in the set

$$\psi(\alpha,k,\nu) = \{x\in[-\rho,\rho], \quad F_2(x)\leqslant y\leqslant F_1(x)\}$$

The proof is similar to that of Theorem 1.

Corollary 2. The global attractor of system (1) is contained in the set

$$\Psi_0 = \bigcap_{\alpha,k,\nu} \Psi(\alpha,k,\nu)$$

We will now examine the van der Pol equation, i.e. the case where

$$E(t)=0, \quad F(y)=y^3/3-y$$

Here, condition (2) is satisfied if

$$k = 2(\alpha+1)^{3/2}/3+\varepsilon \tag{11}$$

where ε is any positive number. Then, assuming that $\alpha = 2/\mu$, we will establish that the set Ω_0 is positioned in the band

$$\{|x|\leqslant M, \quad y\in\mathbb{R}^1\}, \quad M = 2\mu(2/\mu+1)^{3/2}/3 \tag{12}$$

It is also clear that, when $\mu\alpha \geqslant 2$, the following inequalities are satisfied

$$\begin{aligned} &G_1(x,\mu k)\leqslant(\varkappa+\mu k)/(\mu\alpha), \quad \forall x\in[-\varkappa,\mu k] \\ &G_2(x,\mu k)\geqslant-(\varkappa+\mu k)/(\mu\alpha), \quad \forall x\in[-\mu k,\varkappa] \end{aligned} \tag{13}$$

where $\varkappa$ is a certain positive number. Using estimates (12) and (13), we obtain the inclusion

$$\Omega_0 \subset \{|x| \leqslant M, \quad |y| \leqslant N\}, \quad N = (\mu k + M)/(\mu\alpha) \tag{14}$$

The quantity N can be replaced by

$$N = \min_{\alpha \geqslant 2/\mu} 2((\alpha+1)^{3/2} + (2/\mu+1)^{3/2})/(3\alpha) \tag{15}$$

Assuming $\mu \geqslant 2/3$, and selecting $\alpha = 3$. we obtain

$$N = 16/9 + 2(2/\mu+1)^{3/2}/9$$

Examining the case where $\alpha\mu < 2$ and considering condition (11), we obtain

$$\Omega_0 \subset \left\{ |x| \leqslant \frac{2\mu}{3} \min_{\alpha \in (0, 2/\mu)} \left[(\alpha+1)^{3/2} \frac{1+\exp(-\lambda\pi/\omega)}{1-\exp(-\lambda\pi/\omega)} \right] \right.$$
$$\left. |y| \leqslant \frac{4\mu}{3} \min_{\alpha \in (0, 2/\mu)} \left[(\alpha+1)^{3/2} \frac{\sqrt{\lambda^2+\omega^2}\exp(-\lambda\tau(\alpha))}{1-\exp(-\lambda\pi/\omega)} \right] \right\} \tag{16}$$
$$\tau(\alpha) = \frac{1}{\omega}\operatorname{arctg}\frac{\omega}{\lambda}$$

Thus, the limit cycle of the van der Pol equation, which is found in the set Ω_0, can be estimated by means of inclusions (14)–(16).

Applying Theorem 2 with

$$\nu = -\frac{2}{3} - \varepsilon, \quad \gamma = \frac{|R_1|R_2}{1+R_1^2}, \quad R_1 = -\lambda + \sqrt{\lambda^2 - 1}, \quad R_2 = \mu k - \frac{2}{3}\mu$$

where ε is an arbitrary positive number, we obtain

$$|\rho| \leqslant \frac{2}{3}\mu + \frac{|R_1|R_2}{\sqrt{1+R_1^2}}$$

The inclusion

$$\Psi_0 \subset \{|x| \leqslant g(\mu)\}, \quad g(\mu) = \frac{2}{3}\mu + \min_{\alpha \geqslant 2/\mu} \frac{|R_1|R_2}{\sqrt{1+R_1^2}}$$

follows from this.

Thus, for the limit cycle $x(t), y(t)$ of the van der Pol equation considered, the estimate

$$|x(t)| \leqslant g(\mu), \quad \forall t \in \mathbb{R}^1 \tag{17}$$

is satisfied.

We will now examine, for the van der Pol equation, one other comparison system with $\nu = -2/3 - \varepsilon$ and with the parameter γ satisfying the equations

$$\left[(g(\mu) + \tfrac{2}{3}\mu)^2 - \gamma^2\right]^{1/2} + R_2 = \alpha\mu\gamma \quad \text{when} \quad R_2 \leqslant \alpha\mu(g(\mu) + \tfrac{2}{3}\mu) \tag{18}$$

$$\gamma = g(\mu) + \tfrac{2}{3}\mu \quad \text{when} \quad R_2 > \alpha\mu(g(\mu) + \tfrac{2}{3}\mu) \tag{19}$$

The estimate

$$|y(t)| \leqslant \min_{\alpha \geqslant 2/\mu} \gamma, \quad \forall t \in \mathbb{R}^1 \tag{20}$$

follows from relations (17)–(19).

Equation (18) can be written in the form

$$\gamma = \frac{\alpha\mu R_2 + [(1+\alpha^2\mu^2)(g(\mu) + \tfrac{2}{3}\mu)^2 - R_2^2]^{1/2}}{1+\alpha^2\mu^2} \tag{21}$$

Estimate (19)–(21) is more accurate than estimate (14). In particular

$$g(\mu) \leqslant \frac{2}{3}\mu(1+R_3), \quad R_3 = \frac{1}{\sqrt{2}}\left[\left(1+\frac{2}{\mu}\right)^{3/2} - 1\right]$$

Therefore, when $\mu \geqslant 2/3$ and $\alpha = 3$, from Eq. (21) we obtain the estimate

$$\min_{\alpha \geqslant 2/\mu} \gamma \leqslant \frac{\mu^2}{1+9\mu^2}\left[14 + \left\{\left(\frac{4}{9\mu^2} + 4\right)(2+R_3)^2 - \frac{196}{9\mu^2}\right\}^{1/2}\right]$$

Estimates y_{max} of the maximum value $\max|y(t)|$ of the limit cycle with respect to the y coordinate, obtained using inclusions (16) and (20), are given below (when $0.1 \leqslant \mu \leqslant 0.6$, estimate (16) was used, and when $0.7 \leqslant \mu \leqslant 100$, estimate (20) was used; the value of $\max|y(t)|$ was obtained by Odani [13] using a computer experiment):

μ	0.1	0.5	0.6	0.7	1
y_{max}	2.207	2.253	2.273	2.263	2.200
$\max\lvert y(t)\rvert$	2.00010				2.00862
μ	2	5	10	14	100
y_{max}	2.123	2.060	2.032	2.0234	2.003
$\max\lvert y(t)\rvert$	2.01989	2.02151	2.01429		

Note that there are the following estimates of $\max|y(t)|$ for all $\mu \in (0, +\infty)$: $\max|y(t)| < 2.805$ [1], $\max|y(t)| < 2.5425$ [2], and $\max|y(t)| < 2.3439$ [3]. Thus, the results obtained using estimates (16) and (20) are better than these estimates.

Odani [3] put forward the hypothesis that, for any $\mu > 0$, the estimate

$$|y(t)| \leqslant 2.0235$$

holds.

From estimate (20) it follows that this hypothesis is correct for $\mu \geqslant 14$. To prove this correctness for any $\mu > 0$ using the approach proposed here, it is necessary to construct, for $\mu \in (0, 14)$, more complex piecewise-linear comparison systems.

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